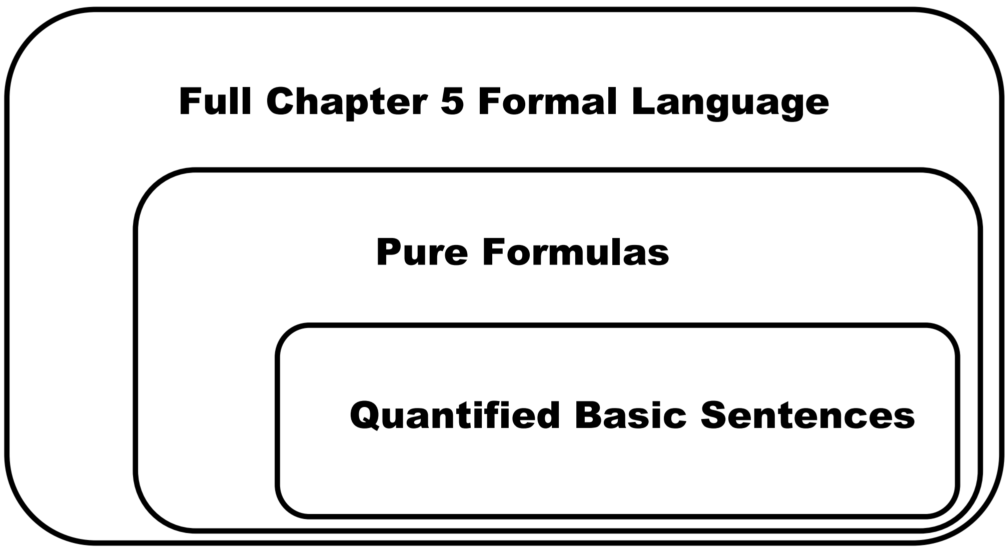
❖ *Pure Formulas* ❖

5.10. Pure Formulas

1. Sentences, Quasi-Sentences, and Formulas. Recall our roadmap of Chapter Five formal languages.



Next in our semantic ascent through these languages is the language of **pure formulas**, where quantifiers are free to attach to strings of symbols far larger than mere variable basics.

The simplest way of presenting this sub-language is by way of the construction rules for the full Chapter Five language. And to set out those rules we return to out earlier discussion of sentences – and non-sentences.

We noted that a little English sentence such as "It is made of wood" doesn't make a complete claim without help from some outside factor – a pointing finger or prominent bit of background context. For I can utter these words and make a true claim (when pointing at a surfboard carved from koa), yet utter the same words to make a false claim in that context (while pointing at a fiberglass and polystyrene surfboard). Since variables are the formal counterpart of pronouns like "it," a formal string such as "Gx" suffers the same incompleteness: without some outside factor pinning down what “x” is pointing to, **“Gx” doesn’t make a complete claim**.

That’s in stark contrast to the atomic sentences encountered earlier. “P” and “Ga” are by themselves capable of being true or false, just like their English counterparts – say, “Exercise is bad for the soul” or “Elvis is a gambler”.

So – taking **sentences** to be complete-claim makers, capable of truth or falsehood – we don’t count the variable atom “Gx” as a formal sentence. But we recognize its close resemblance to genuine sentences: it is, construction-wise, built the same as name atoms (just a predicate-letter-plus-*variable*, rather than predicate-letter-plus-*name-letter*). Coining a new bit of jargon, we say such an incomplete-claim-maker is a **quasi-sentence**.[1]

Just adding that new label doesn’t make clear what counts as a (complete) formal sentence and what’s a (mere) quasi- sentence. But even in advance of precise criteria, a further bit of jargon suffices to complete construction rules for the expanded formal language.

We’ll use “**formula**” as an umbrella term covering any formal **sentence or quasi-sentence**. So “P,” “Ga,” and “Gx” are all formulas.

Formulas

Formal Sentences **Quasi-Sentences**

P, Ga Gx

The revised construction rules for the full Chapter Five language can then be stated in terms of formulas.

[1] Adapting the “quasi-statement” of (Lambert and van Fraassen 1972: 79-80) – though attaching a different meaning to the phrase.

Construction Rules (*For Full Ch. 5 Language*)

Terms

T1. Name letters are terms
T2. Variables are terms

Atomic Formulas:

A1. Sentence letters are atomic formulas
A2. A predicate letter followed by a term is an atomic formula.

Formulas:

1. Atomic formulas are formulas.
2. If ● is a formula, then ~● is a formula.
3. If ● and ▲ are formulas, then (● ∧ ▲) is a formula.
4. If ● and ▲ are formulas, then (● ∨ ▲) is a formula.
5. If ● and ▲ are formulas, then (● →▲) is a formula.
6. If ● and ▲ are formulas, then (● ↔▲) is a formula.

7a. If ★ is a variable and ● is a formula, then ∃★ ● is a formula.
7b. If ★ is a variable and ● is a formula, then ∀★ ● is a formula.

2. Pure Formulas. Note that since a quantifier attaches to the left of a formula, **construction-wise** quantifiers act just like tildes.

In Chapter Two we called the sentence which the tilde attaches to the **scope** of that tilde.[2] Here likewise: the formula which a quantifier attaches to is the **scope** of that quantifier.

So in the formula "∀x Gx", the scope of "∀x" is the formula "Gx".

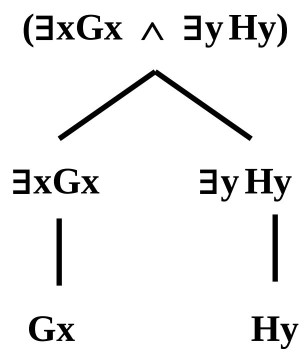
∀x Gx *(7b)*
|
Gx *(T2, A2, 1)*

[2] In 2.10.

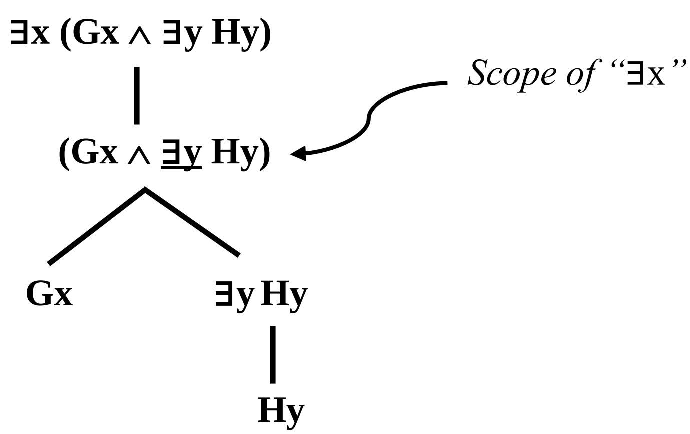
From there it’s easy to set out the pure formulas (and pure sentences) through a simple limit on quantifier scope: **a formula is pure if no quantifier shows up within the scope of another quantifier**.

> **Pure formula**: a formula where no quantifier appears within the scope of another quantifier[3]

So “**(∃xGx ∧ ∃y Hy)**” **is a pure formula**, because though it has two quantifiers, neither appears within the scope of the other. (The scope of “**∃xGx**” is just “Gx,” while the scope of “**∃y Hy**” is “Hy”.)



But “**∃x (Gx ∧ ∃y Hy)**” **isn’t a pure formula**, because the quantifier “**∃y**” appears within the scope of “**∃x**”.



[3] Adopting “pure” from (Quine 1982: 152).

Scope overlap provides an equally quick way of understanding pure formulas. For note that in the pure formula “**(∃xGx ∧ ∃y Hy)**” the scopes of “**∃x**” and “**∃y**” don’t overlap.

(∃xGx ∧ ∃y Hy)

Scope of “∃y”

Scope of “∃x”

Whereas in the impure formula “**∃x (Gx ∧ ∃y Hy)**” the scopes of “**∃x**” and “**∃x**” do overlap.

∃x (Gx → **∃y** ~**Gy**)

Scope of “∃y”

Scope of “∃x”

So all the following are pure formulas.

∀x (Gx ∧ Hy)   (∀x Gx → (Gw ∨ ∃y Jy))
(∃z (Gz → Hx) ∧ ∀x (Hx ∧ Ix))

But none of the following are pure formulas.

∀x (Gx ∧ ∃y Hy)   ∃z (Gz → ∀x (Hx ∧ Ix))
∀x ∃y (Gx ∧ Hy)   (∀x Gx → ∀w (Gw ∨ ∃y Jy))

Since this is the only requirement for a Chapter Five formula to count as a pure formula, a small change to our construction rules suffices to stake out the pure formulas: the rules introducing quantifiers must be restricted to attach quantifiers only to formulas not already containing a quantifier.

> P7a. If ★ is a variable and ● is a **formula containing no quantifiers**, then ∃★ ● is a pure formula.
>
> P7b. If ★ is a variable and ● is a **formula containing no quantifiers**, then ∀★ ● is a pure formula.[4]

Restricting quantifier scope in this way might seem to drastically constrain use of the full formal language; but that turns out not to be true in practice. For the sentences (and formulas) exhibiting the forbidden quantifier overlap are typically the ones whose interpretation tends to puzzle us. Consider again the pure and impure formulas we looked at earlier.

> Pure formula: **(∃xGx ∧ ∃y Hy)**
>
> Impure formula: **∃x (Gx ∧ ∃y Hy)**

If "G" translates "is a waitress" and "H" translates "is a gambler," the pure formula makes the straightforward claim that someone's a waitress and someone's a gambler. By contrast, the impure formula says there's someone with the following feature: *being-a-waitress-and-someone-being-a-gambler*. That unorthodox feature is less likely to crop up in everyday conversation;[5] and not coincidentally, translation of English sentences into the Chapter Five formal language won't usually call for impure formulas.[6]

While we can make sense of impure formulas such as the above example, it turns out the simplest way of doing so is to translate into an equivalent pure formula.[7] And it's a happy feature of pure formulas that we can always find such a translation: as we'll later show, **any formula of the full Chapter Five language is equivalent to a pure formula**.[8] (In the terminology of earlier chapters: the language of pure formulas and the full Chapter Five

[4] Note that **quantified basic sentences all count as pure formulas**, since the quantifier in a quantified basic never has another quantifier in its scope.

[5] So the impure sentence says *there's a waitress who has the following feature: someone (not necessarily that waitress) is a gambler*.

[6] The formal language of Chapter Six, by contrast, makes unavoidable use of impure formulas.

[7] So, for instance, the above impure formula about waitresses and gamblers turns out to be equivalent to the pure formula on the same topic directly above it.

[8] In 5.X.

language are **expressively equivalent**.) So we lose no expressive power (and spare ourselves considerable semantic head-scratching) by limiting ourselves to pure formulas.[9]

But before we can apply the language of pure formulas to translating English claims, we must return to the topic of sentences vs. quasi-sentences, and draw a clear line between the two.

[9] And as we'll see (in 5.X), the language of pure formulas also allows us to use a simpler account of "instance" than is required for the full Chapter Five language – simplifying in turn the truth tree and deductive rules using such instances.

Construction Rules (for Pure Formulas)

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P7b. If ★ is a variable and ● is a **formula containing no quantifiers**, then ∀★ ● is a pure formula.